Isomonodromy and Painlevé type equations. Search and Case studies.

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1. Introduction, PP, moving poles

Every solution of a *linear* differential equation over $\mathbb{C}(z)$, e.g.,

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admits analytic continuation outside the singular points.

This property for ordinary *nonlinear* differential equations over $\mathbb{C}(z)$ has the name Painlevé property (PP) and can be formulated as:

there is a finite set $S \subseteq \mathbb{C} \cup \{\infty\}$ such that any local solution admits an analytic continuation involving poles outside the set S. The poles can be anywhere and are called moving poles.

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Questions:

- Is every (nonlinear) differential equation with PP induced by isomonodromy?
- How to produce isomonodromic families?

3. The Riemann–Hilbert method

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The given data also prescribe the possibilities for topological monodromy, Stokes matrices and links.
This defines a space $\mathcal{R}$ of analytic data.

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Last step: explicit computation of this differential equations by means of what is called a Lax pair.

5. A quick look at singularities

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M is "regular singular" if A can be chosen to be constant.
Formally, i.e., over a finite extension of $\mathbb{C}((z^{-1}))$, one can write M as direct sum of modules represented by operators of the form (here size 4)

$$z\frac{d}{dz} + \begin{pmatrix} q+a & 0 & 0 & 0 \\ 1 & q+a & 0 & 0 \\ 0 & 1 & q+a & 0 \\ 0 & 0 & 1 & q+a \end{pmatrix}, \quad q \in z^{1/m}\mathbb{C}[z^{1/m}], \quad a \in \mathbb{C}.$$

The q's are called eigenvalues.
The Katz invariant is $\max_q \deg_z(q)$.
The formal monodromy sends $z^{1/m}$ to $e^{2\pi i/m}z^{1/m}$ and acts on the decomposition of M.

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Property: $\gamma_V(V_q) = V_{\gamma q}$, where γ is the automorphism of the algebraic closure of $\mathbb{C}((z^{-1}))$ given by $\gamma(z^\lambda) = e^{2\pi i \lambda} z^\lambda$.

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One has singular directions $d_1 < d_2 < \cdots < d_r \in [0, 1) = \mathbb{R}/2\pi\mathbb{Z}$ having the properties: $St_{d+1} = \gamma_V^{-1} St_d \gamma_V$; $St_d = id$ for $d \notin \{d_1, d_2, \ldots, d_r\} + \mathbb{Z}$; St_d has a special form and the monodromy identity: $mon_\infty = \gamma_V \circ St_{d_r} \circ \cdots \circ St_{d_1}$ with mon_∞ is the monodromy around $z = \infty$.

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(b). $(y')^2 = a(y^3 + by + c)$ with $b, c \in \mathbb{C}$, $a \in \mathbb{C}(z)$ (Weierstrass),
(c). $F(y', y, z) = 0$ equivalent to $\frac{dy}{dz} = 0$ after a finite extension of $\mathbb{C}(z)$.

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In matrix form $\delta + A = \delta + \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, where δ is the derivation under consideration. For a family of matrix Risch differential operators $\delta + A$, parametrized by a variable t, the existence of a Lax pair $\{\delta + A, \frac{d}{dt} + B\}$ is equivalent to $B = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$ and the equations $\frac{d}{dt}(a) = \delta(c)$ and $\frac{d}{dt}(b) = \delta(d) + ad - bc$.

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(c) The monodromy of a module $M \in \mathcal{M}$ consists (by definition) of the topological monodromies for chosen generators of the fundamental group of $X \setminus S$ and the Stokes matrices at the irregular singular points. These data form an element in G^r for some r where $G = \{\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} | \alpha, \beta \in \mathbb{C}, \alpha \neq 0\}$.

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Recall requirement:

the fibres of the surjective RH : $\mathcal{M} \to \mathcal{R}$ have dimension 1.

9. Finding the families with properties (a)–(c).

The (connected components of the) fibres of *RH* are parametrized by what *RH* forgets, i.e., the position of the points S and the coefficients of the eigenvalue q (if s is irregular). The cases are classified modulo the action of PGL_2 on $\mathbb{P}^1$. Hence S has at most 4 points.

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(i). In case S has four points, these can be chosen to be $0, 1, \infty, t$. The (connected components of the) fibres of *RH* are parametrized by t and all the singularities are regular singular. Thus the 1-dimensional submodule F is given by $\frac{d}{dz} + a$ with $a = \frac{a_0}{z} + \frac{a_1}{z-1} + \frac{a_t}{z-t}$ with constants a_0, a_1, a_t. The group S_3 of the automorphisms of $\mathbb{P}^1$ permuting $\{0, 1, \infty\}$, also permutes the various $\frac{d}{dz} + a$.

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Similar arguments give *the list for the 1-dimensional submodules F*:

(i). $\frac{d}{dz} + \frac{a_0}{z} + \frac{a_1}{z-1} + \frac{a_t}{z-t}$. $S = \{0, 1, \infty, t\}$. Related to P_6.
(ii). $\frac{d}{dz} + \frac{a_0}{z} + \frac{a_1}{z-1} + t$. $S = \{0, 1, \infty\}$, ∞ Katz invariant 1. Related to P_5.
(iii). $z\frac{d}{dz} + a_0 + tz + z^2$. $S = \{0, \infty\}$, ∞ Katz invariant 2. Related to P_4.
(iv). $z\frac{d}{dz} + \frac{t}{z} + a_0 + z$. $S = \{0, \infty\}$, both Katz invariant 1. Related to P_3.
(v). $\frac{d}{dz} + t + z^2$. $S = \{\infty\}$ with Katz invariant 3. Related to P_2.

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For each of the cases (i)–(v), we computed an operator representing a general $M \in \mathcal{M}$, the corresponding data in $\mathcal{R}$, the Lax pair, and finally the resulting nonlinear first order equation.

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(i) $f' = \frac{a_0}{t} f^2 + (\frac{a_0 + a_2}{t} - \frac{a_1 + a_2}{t-1}) f - \frac{a_1}{t-1}$. The Riccati equation of the hypergeometric equation ${}_2F_1(a_2, a_0 + a_1 + a_2, 1 - a_0 - a_2; t)$.
(ii) $f' = -\frac{a_1}{t} f^2 + (\frac{a_0 + a_1 + 1}{t} + 1) f - 1$. The Riccati equation of Kummer's confluent hypergeometric equation ${}_1F_1(a, c; z)$.
(iii) $b_1' + a_0 b_1^2 - t b_1 + 1 = 0$. The Riccati equation of the parabolic cylinder functions.
(iv) $t b_1' + 1 + (1 - a_0) b_1 + t b_1^2 = 0$. The Riccati equation of the Bessel equation.
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It is not surprising that these equations are in fact equivalent to the reducible locus of the Painlevé equations $P_6 - P_2$. We now give some details for case (ii).

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On the open subset $x \neq 0$, one normalizes x to 1, by base change. After that, the basis of $V(\infty)$ is unique up to multiplication of the base vectors by the same scalar.

12. case (ii), $\frac{d}{dz}+\frac{a_0}{z}+\frac{a_1}{z-1}+t$, $S=\{0,1,\infty\}$, ∞ Katz invariant 1. $\mathcal{R}$

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It follows that $\mathcal{R}$ has dimension 3 and the parameter space $\mathcal{P}$ (represented by the variables g_0, g_1) has dimension 2.

13. case (ii), $\frac{d}{dz}+\frac{a_0}{z}+\frac{a_1}{z-1}+t$, $S=\{0,1,\infty\}$, ∞ Katz invariant 1. Lax pair

A good choice for the Lax pair is

$$\frac{d}{dz}+\begin{pmatrix} \frac{a_0}{z}+\frac{a_1}{z-1}+t & \frac{b_0}{z}+\frac{b_1}{z-1}+b_2 \\ 0 & 0 \end{pmatrix} \text{ and}$$

$$\frac{d}{dt}+\begin{pmatrix} \frac{c_0}{z}+\frac{c_1}{z-1}+c_2 z & \frac{d_0}{z}+\frac{d_1}{z-1}+d_2 z \\ 0 & 0 \end{pmatrix}.$$

13. case (ii), $\frac{d}{dz} + \frac{a_0}{z} + \frac{a_1}{z-1} + t$, $S = \{0, 1, \infty\}$, ∞ Katz invariant 1. Lax pair

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$$\frac{d}{dt} + \begin{pmatrix} \frac{c_0}{z} + \frac{c_1}{z-1} + c_2 z & \frac{d_0}{z} + \frac{d_1}{z-1} + d_2 z \\ 0 & 0 \end{pmatrix}.$$ This yields

$$b_0' = 0, \; b_1' = \frac{-tb_1 + a_1 b_2}{t}, \; b_2' = \frac{-tb_0 - tb_1 + (a_0 + a_2 + 1)b_2}{t}.$$

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$\frac{d}{dt} + \begin{pmatrix} \frac{c_0}{z} + \frac{c_1}{z-1} + c_2 z & \frac{d_0}{z} + \frac{d_1}{z-1} + d_2 z \\ 0 & 0 \end{pmatrix}$. This yields

$$b_0' = 0,\ \ b_1' = \frac{-tb_1 + a_1 b_2}{t},\ \ b_2' = \frac{-tb_0 - tb_1 + (a_0 + a_2 + 1)b_2}{t}.$$

After normalization to $b_0 = 0$ one obtains for $\binom{b_1}{b_2}$ the matrix differential equation $\frac{d}{dt} + \begin{pmatrix} 1 & 1 \\ -\frac{a_1}{t} & -\frac{a_0+a_1+1}{t} \end{pmatrix}$. This is a matrix differential equation for Kummer's confluent hypergeometric equation ${}_1F_1(a, c; z)$.

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$\frac{d}{dz}+\begin{pmatrix} \frac{a_0}{z}+\frac{a_1}{z-1}+t & \frac{b_0}{z}+\frac{b_1}{z-1}+b_2 \\ 0 & 0 \end{pmatrix}$ and

$\frac{d}{dt}+\begin{pmatrix} \frac{c_0}{z}+\frac{c_1}{z-1}+c_2 z & \frac{d_0}{z}+\frac{d_1}{z-1}+d_2 z \\ 0 & 0 \end{pmatrix}$. This yields

$$b_0'=0,\ b_1'=\frac{-tb_1+a_1b_2}{t},\ b_2'=\frac{-tb_0-tb_1+(a_0+a_2+1)b_2}{t}.$$

After normalization to $b_0=0$ one obtains for $\binom{b_1}{b_2}$ the matrix differential equation $\frac{d}{dt}+\begin{pmatrix} 1 & 1 \\ -\frac{a_1}{t} & -\frac{a_0+a_1+1}{t} \end{pmatrix}$. This is a matrix differential equation for Kummer's confluent hypergeometric equation ${}_1F_1(a,c;z)$. The first order differential equation for $f=\frac{b_2}{b_1}$ obtained from isomonodromy is

$$f'=-\frac{a_1}{t}f^2+(\frac{a_0+a_1+1}{t}+1)f-1.$$

14. More first order equations from isomonodromy?

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14. More first order equations from isomonodromy?

Note: the above cases of first order equations induced by isomonodromy came from subfamilies of reducible families of connections of rank two on $\mathbb{P}^1$. We describe below an exceptional case of subfamilies of a reducible family of connections $\mathcal{M}$ of rank three on $\mathbb{P}^1$. Our two questions remain unanswered.

$\mathcal{M}$ is the moduli space of connections on the free bundle of rank 3 on $\mathbb{P}^1$ which is induced by the set of differential modules M over $\mathbb{C}(z)$ defined by the conditions:
(a). $\dim M = 3$, $\Lambda^3 M = 1$, singular points $z = 0$ and $z = \infty$,
(b). $z = 0$ is regular singular and $z = \infty$ is irregular singular and has eigenvalues $z, tz, (-1-t)z$.
This moduli space has dimension 7 (counting t as variable).

15. A differential operator for $\mathcal{M}$

It can be shown that the matrix differential operator

$$z\frac{d}{dz} + \begin{pmatrix} z + a_0 & v_1 & v_2 \\ 1 & tz + a_1 & 1 \\ v_3 & v_4 & (-1-t)z - a_0 - a_1 \end{pmatrix}$$

represents a Zariski open affine, dense subset of $\mathcal{M}$. This operator (with $v_1, \ldots, v_4$ as functions of t; the a_0, a_1 are parameters) is completed to a Lax pair with the operator $\frac{d}{dt} + B_0(t) + zB_1(t)$. There are explicit formulas for $v_1', \ldots, v_4'$.

One observes that the differential operator has three reducible subfamilies of $\mathcal{M}$, namely given by

16. reducible subfamilies

(i) $v_3 = v_4 = 0$, (ii) $v_1 = v_2 = 0$, (iii) $v_1 = v_4 = 0$. The differential equations for these families are:

$$\text{(i)}\ v_1' = 0 \text{ and } v_2' = \frac{3(2t+1)v_2^2 - 3(t+2)v_1 + 9(-a_0 t + a_1)v_2}{(t-1)(2t+1)(t+2)}.$$

$$\text{(ii)}\ v_4' = 0 \text{ and } v_3' = \frac{3(t-1)v_3^2 + 9(a_0 - a_1)v_3 + 3(t+2)v_4}{(t-1)(2t+1)(t+2)}.$$

$$\text{(iii)}\ v_2' = \frac{(6t+3)v_2^2 - 3v_3(t-1)v_2 - 3v_1(t+2) + (-9a_0 t + 9a_1)v_2}{(t-1)(2t+1)(t+2)} \quad \text{and}$$

$$v_3' = \frac{(3t-3)v_3^2 + (-6t-3)v_2 v_3 + (9a_0 t - 9a_1)v_3}{(t-1)(2t+1)(t+2)}.$$

(i) and (ii) are examples of Riccati equations obtained by isomonodromy. In case (iii), the term $v_2 v_3$ is a parameter (and thus a constant). Therefore the two equations are "equivalent" Riccati equations.

17. Case studies. The Painlevé equations

Each of the equations $P_1 - P_5$ is derived from a family $RH : \mathcal{M} \to \mathcal{R}$. Example: Painlevé P_1.
$\mathcal{M}$ defined by $\dim M = 2$, $\Lambda^2 M$ trivial, the only singularity is ∞ and has Katz invariant $5/2$. The eigenvalues at ∞ are $\pm(*z^{5/2} + *z^{3/2} + *z^{1/2})$ and are normalized by the transformation $z \mapsto az + b$ to $\pm(z^{5/2} + tz^{1/2})$.

The monodromy space $\mathcal{R}$ is the space of the Stokes matrices. There are 5 singular directions $\frac{j}{5}, 0 \leq j \leq 4$ and the trivial topological monodromy equals
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_5 & 1 \end{pmatrix}\begin{pmatrix} 1 & x_4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_3 & 1 \end{pmatrix}\begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}$ and so $\dim \mathcal{R} = 2$.
$\mathcal{R}$ turns out to be an affine non singular cubic surface with three lines at infinity.

18. Continuation of Painlevé 1

The fibres of $RH : \mathcal{M} \to \mathcal{R}$ are parametrized by t. A Zariski open part of $\mathcal{M}$ is represented by the matrix differential operator $\frac{d}{dz} + \begin{pmatrix} p & t + q^2 + qz + z^2 \\ z - q & -p \end{pmatrix}$.

This is completed to a Lax pair by $\frac{d}{dt} + \begin{pmatrix} 0 & 2q+z \\ 1 & 0 \end{pmatrix}$.

One obtains the equations $\frac{dq}{dt} = 2p$, $\frac{dp}{dt} = 3q^2 + t$ and finally $q'' = 6q^2 + 2t$, the first Painlevé equation.

19. A new isomonodromic family of dimension 2; a companion of P_1

$\mathcal{M}$ is defined by: determinant trivial; regular singular at $z = 0$; eigenvalues $\pm(z^{5/2} + \frac{t}{2} z^{1/2})$ at $z = \infty$.
$\mathcal{R} \cong \mathbb{C}^5$ (again 5 Stokes matrices, no relations).
The fibres of $\mathcal{M} \to \mathcal{R}$ are parametrized by t.
$\mathcal{R} \to \mathcal{P} \cong \mathbb{C}=$ the parameterspace = the characteristic polynomial of the monodromy matrix at $z = 0$.
$z\frac{d}{dz} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^3 + \begin{pmatrix} 0 & b_2 \\ 1 & 0 \end{pmatrix} z^2 + \begin{pmatrix} a_1 & b_1 \\ -b_2 & -a_1 \end{pmatrix} z + \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix}$:
$t = b_1 - b_2^2 + c_0$, parameter $p_0 = a_0^2 + b_0 c_0$.
the Lax pair yields the Painelevé type vector field :
$a_0' = 2b_2 c_0 - \frac{p_0 - a_0^2}{c_0}$, $c_0' = 2a_0$, $a_1' = -3b_2^2 + 2c_0 - t$, $b_2' = -2a_1$.

20. Continuation of a companion of P_1

$c_0 = 0$ leads to P_1. Assume $c_0 \neq 0$. After elimination only $f := b_2$ and its derivatives $f_j := (\frac{d}{dt})^j b_2,\ j = 1, 2, 3, 4$ survive and there is one relation:

$$-2(6f^2 - f_2 + 2t)f_4 = 288f^5 - 240f^3 f_2 + 192tf^3 - 24ff_1 f_3 + 32ff_2^2 - 80tff_2$$

$$+32ft^2 + 24f_1^2 f_2 - 48tf_1^2 + 48ff_1 + f_3^2 - 4f_3 + 64p_0 + 4$$

Note that the denominator of $f_4 := b_2^{(4)}$ is the equation for P_1.
This Painlevé type equation is explicit of order four.

21. The full companion of P_1

The previous example "companion of P_1" is not natural in some sense. Natural conditions on the modules $M \in \mathcal{M}$ are:
$\dim M = 2,\ \Lambda^2 M = 1,\ z = 0$ regular singular,
$z = \infty$ irregular and Katz invariant $\frac{5}{2}$.
It follows that the eigenvalues are $\pm(z^{5/2} + \frac{t_1}{2} z^{3/2} + \frac{t_2}{2} z^{1/2})$.
Now $RH : \mathcal{M} \to \mathcal{R}$ forgets the two "time variables" t_1, t_2.

Now isomonodromy and Lax pairs in two variables t_1, t_2. The differential operator $z\frac{d}{dz} + A(z, t_1, t_2)$ commutes with two operators $\frac{d}{dt_j} + B_j,\ j = 1, 2$. The family of dimension 2+5 depends on t_1, t_2 and a_0, a_1, b_0, b_1, b_2 variables. With the notation $df = \frac{df}{dt_1} dt_1 + \frac{d}{dt_2} dt_2$ the Painlevé equations are:

22. Painlevé equations for the full companion of P_1

$$d(a_0) = \frac{1}{48}\{16b_2^4 - 16b_2^3 t_1 + 4b_2 t_1^3 - t_1^4 - 48b_1 b_2^2 + 32b_1 b_2 t_1 - 4b_1 t_1^2 + 32b_2^2 t_2 - 16b_2 t_1 t_2 - 16b_0 b_2 +$$

$$8b_0 t_1 + 32b_1^2 - 48b_1 t_2 + 16t_2^2\}dt_1 + \{-2b_2^3 + 3b_2^2 t_1 - \frac{3b_2 t_1^2}{2} + \frac{t_1^3}{4} + 2b_1 b_2 - b_1 t_1 - 2b_2 t_2 + t_1 t_2 + b_0\}dt_2$$

$$d(a_1) = \frac{1}{24}\{-16b_2^3 + 20b_2^2 t_1 - 4b_2 t_1^2 - t_1^3 + 16b_1 b_2 - 16b_1 t_1 - 16b_2 t_2 + 12t_1 t_2 + 8b_0\}dt_1$$

$$+\{b_2^2 - 2b_2 t_1 + 3/4(t_1^2) + 2b_1 - t_2\}dt_2$$

$$d(b_0) = \frac{1}{6}\{-4a_0 b_2^2 + a_0 t_1^2 + 8a_0 b_1 - 4a_0 t_2 - 4a_1 b_0\}dt_1 + \{(4b_2 - 2t_1)a_0\}dt_2$$

$$d(b_1) = \frac{1}{6}\{-4a_1 b_2^2 + a_1 t_1^2 + 4a_0 b_2 - 2a_0 t_1 + 4a_1 b_1 - 4a_1 t_2 + 2b_2 - t_1\}dt_1 + \{4a_1 b_2 - 2a_1 t_1 + 2a_0 + 1\}dt_2$$

$$d(b_2) = \frac{1}{3}\{-a_1 t_1 + 2a_0 + 2\}dt_1 + 2a_1 dt_2.$$

with parameter=$p_0 = a_0^2 + b_0 c_0$.

23. A new hierarchy $\mathcal{M}_n$, related to $P_3(D_8)$

For any $n \geq 2$, $\mathcal{M}_n$ is defined by the differential modules M over $\mathbb{C}(z)$ given by:
(i) $\dim M = n$, $\Lambda^n M = 1$, i.e., the trivial differential module,
(ii) the only singularities are $z = 0$ and $z = \infty$; they are both irregular singular, totally ramified and have Katz invariant $\frac{1}{n}$.

Part (ii) is made explicit by requiring that $z^{-1/n}$ and its conjugates are the eigenvalues at $z = 0$ and $t^{1/n} z^{1/n}$ with $t \in \mathbb{C}^*$ and its conjugates are the eigenvalues at $z = \infty$.

A direct computation of a differential operator for $\mathcal{M}_n$ seems hardly possible. Therefore we do a trick.

24. Constructing the connection by symmetry

The operator $D := z\frac{d}{dz} + A$ of size $n \times n$ over $\mathbb{C}(z)$, that we try to construct, is seen as operator on a vector space V of dimension n over $\mathbb{C}(z)$. The extension of D to $W := \mathbb{C}(z^{1/n}) \otimes V$, also called D, has no ramification.

γ is the automorphism of $\mathbb{C}(z^{1/n})/\mathbb{C}(z)$ with $\gamma(z^{1/n}) = e^{2\pi i/n}z^{1/n}$. $\sigma : W \to W$ is the semi-linear map with $\sigma(f \otimes v) = \gamma(f) \otimes v$. Define the trace $tr : W \to V$ by $tr(w) = \sum_{j=0}^{n-1} \sigma^j(w)$. We expect the following :

There is a basis $e_0, e_1, \ldots, e_{n-1}$ of W such that σ acts as $e_0 \mapsto e_1 \mapsto \cdots \mapsto e_{n-1} \mapsto e_0$ and D has on this basis only poles of order 1 at $z^{1/n} = 0$ and at $z^{1/n} = \infty$. Since $\sigma D = D\sigma$, $D(e_0)$ determines D and $D(e_0)$ has the form $\sum_{j=0}^{n-1}(a_j z^{-1/n} + b_j + c_j z^{1/n})e_j$ with $a_j, b_j, c_j \in \mathbb{C}$.

25. D on basis $B_0, \ldots, B_{n-2}, z^{-1}B_{n-1}$ of invariants

From the basis $e_0, \ldots, e_{n-1}$ one constructs a σ-invariant basis of V, namely $B_0, B_1, \ldots, B_{n-1}$ by $B_j = tr(z^{j/n}e_0)$ for $j = 0, \ldots, n-1$. The given data for $D(e_0)$ induces a formula $z\frac{d}{dz} + A$ for D on the basis $B_0, \ldots, B_{n-2}, z^{-1}B_{n-1}$. There is a normalization
$D(e_0) = (z^{-1/n} + b_0 + c_0 z^{1/n})e_0 + \sum_{j=1}^{n-1}(b_j + c_j z^{1/n})e_j$,
$b_0 = \frac{3-n}{2n}$, $\beta = t$. The operator E commuting with D, is σ-invariant and is determined by $E(e_0) = z^{1/n}\sum_{j=0}^{n-1} c_j e_j$. Now we skip many details of the construction which involves also a computation of the monodromy space $\mathcal{R}$.

26. explicit Lax pair and Painlevé type equations

For general n, the Lax pair is

$$z\frac{d}{dz}+\begin{pmatrix} d_0 & 1 & 0 & . & 0 & f_0 \\ f_1 & d_1 & 1 & . & 0 & 0 \\ 0 & f_2 & d_2 & . & 0 & 0 \\ . & . & . & . & 1 & . \\ 0 & . & . & f_{n-2} & d_{n-2} & \frac{1}{z} \\ 1 & 0 & . & 0 & f_{n-1}z & d_{n-1} \end{pmatrix},\ t\frac{d}{dt}+\begin{pmatrix} 0 & 0 & 0 & . & 0 & f_0 \\ f_1 & 0 & 0 & . & 0 & 0 \\ 0 & f_2 & 0 & . & 0 & 0 \\ . & . & . & . & 0 & . \\ 0 & . & . & f_{n-2} & 0 & 0 \\ 0 & 0 & . & 0 & f_{n-1}z & 0 \end{pmatrix}$$

with $\sum d_j = 0,\ \prod f_j = t$. The Painlevé type equations are

$$t\frac{f_0'}{f_0} = d_0 - d_{n-1},\ t\frac{f_1'}{f_1} = d_1 - d_0, \cdots, t\frac{f_{n-1}'}{f_{n-1}} = d_{n-1} - d_{n-2} + 1,$$

$$td_0' = f_1 - f_0,\ td_1' = f_2 - f_1, \cdots\cdots, td_{n-1}' = f_0 - f_{n-1}.$$

27. The case $n = 2$ identifies with $P_3(D_8)$

The definition of the family of connections $\mathcal{M}_2$ coincides with the well known isomonodromic family for $P_3(D_8)$. The Lax pair is $z\frac{d}{dz} + \begin{pmatrix} d_0 & \frac{1}{z} + f_0 \\ 1 + f_1 z & d_1 \end{pmatrix}$, $t\frac{d}{dt} + \begin{pmatrix} 0 & f_0 \\ f_1 z & 0 \end{pmatrix}$ with $f_0 f_1 = t$ and $d_0 + d_1 = 0$. The equations are

$$t\frac{f_0'}{f_0} = d_1 - d_0,\ t\frac{f_1'}{f_1} = d_0 - d_1 + 1,\ td_0' = f_1 - f_0,\ td_1' = f_0 - f_1.$$

With the normalization $f_1 = -1$ one obtains the standard formulas.

28. The Noumi-Yamada hierarchy revisited

For $n \geq 3$ one considers a moduli space $\mathcal{M}_n$ corresponding to differential modules M over $\mathbb{C}(z)$ with the properties:
(i). $\dim M = n$, $\Lambda^n M = 1$. The singular points of M are $z = 0$ and $z = \infty$.
(ii). $z = 0$ is a regular singular point
(iii). $z = \infty$ is irregular, totally ramified and Katz invariant $\frac{2}{n}$.
This implies that $z^{2/n} + tz^{1/n}$ and its conjugates are the eigenvalues. $t \in \mathbb{C}$ for odd n, $t \in \mathbb{C}^*$ for even n.
Choices of lattices at $z = 0$ and $z = \infty$ are needed to assure the existence of a moduli space.

A direct approach to compute a matrix differential operator D seems hopeless. However, as before, one can make a guess for the form of the operator D on a vector space over $\mathbb{C}(z^{1/n})$.

29. The symmetric approach for D

This approach leads to the Lax pair of Noumi and Yamada.
Let $e_0, \ldots, e_{n-1}$ be a basis of this vector space and let σ denote the semi-linear automorphism of this vector space such that $\sigma : e_0 \mapsto e_1 \mapsto \cdots \mapsto e_{n-1} \mapsto e_0$. Define the σ invariant operator D by the formula

$$D(e_0) = (z^{2/n} + tz^{1/n})e_0 + \sum_{i=1}^{n-1}(a_i + b_i z^{1/n})e_i.$$

For the operator $E := \frac{d}{dt} + B$ such that $\{D, E\}$ forms a Lax pair, one makes the guess that E is the σ-invariant operator with $E(e_0) = z^{1/n}e_0 + \sum_{j=1}^{n-1} c_j e_j$. Put $\omega = e^{2\pi i/n}$.

30. Formulas for D and E

One deduces from this the matrix of D with respect to the basis $B_0, \ldots, B_{n-1}$ with $B_j := \sum_{k=0}^{n-1} \sigma^k(z^{j/n}e_0)$ for $0 \leq j \leq n-1$ and $B_n := zB_0, B_{n+1} := zB_1$. The formula is

$$D(B_j) = \frac{j}{n}B_j + \sum_{i=1}^{n-1} a_i\omega^{-ij}B_j + tB_{j+1} + \sum_{i=1}^{n-1} b_i\omega^{-i(j+1)}B_{j+1} + B_{j+2},$$

with $\omega = e^{2\pi i/n}$. The formula for E on this basis is

$$E(B_j) = B_{j+1} + (\sum_{k=1}^{n-1} \omega^{-kj}c_k)B_j.$$

31. D and the topological monodromy

The operator D is

$$z\frac{d}{dz}+\begin{pmatrix} \epsilon_0 & 0 & 0 & * & * & z & zf_0 \\ f_1 & \epsilon_1 & 0 & 0 & * & 0 & z \\ 1 & f_2 & \epsilon_2 & 0 & * & * & 0 \\ 0 & 1 & f_3 & \epsilon_3 & 0 & * & * \\ * & * & * & * & * & * & * \\ * & * & * & 1 & f_{n-2} & \epsilon_{n-2} & 0 \\ * & * & * & 0 & 1 & f_{n-1} & \epsilon_{n-1} \end{pmatrix};\ \epsilon_j=\frac{j}{n}+\sum_{i=1}^{n-1}a_i\omega^{-ij};$$

$f_j = t+\sum_{i=1}^{n-1} b_i\omega^{-ij}$.

$\sum \epsilon_j = \frac{n-1}{2}$ and $\sum f_j = nt$. The $\epsilon_0,\ldots,\epsilon_{n-1}$ are the parameters of the family. The $\{e^{2\pi i\epsilon_j}\}$ are the eigenvalues of the topological monodromy at $z=0$. For an isomonodromic family the ϵ_j are constant and the $f_0,\ldots,f_{n-1}$ are analytic functions of the parameter t.

32. Matrix form and Painlevé type equations

$$E = \frac{d}{dt} + \begin{pmatrix} g_0 & 0 & 0 & 0 & * & * & z \\ 1 & g_1 & 0 & 0 & * & * & 0 \\ 0 & 1 & g_2 & 0 & * & * & 0 \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & g_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & g_{n-1} \end{pmatrix}, \; g_j = (\sum_{k=1}^{n-1} \omega^{-kj} c_k), \sum g_j = 0.$$

For an isomonodromic family, the $\{g_j\}$ are functions of t and are in fact eliminated by the Lax pair condition $DE = ED$. For $n = 5$, the Painlevé type differential equations for this Lax pair are

$$f_1' = f_1(-f_1 - 2f_2 - 2f_4 + t) + 2\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$$

$$f_2' = f_2(-2f_1 + f_2 - 2f_4 - t) - \epsilon_1 + \epsilon_2$$

$$f_3' = f_3(-2f_1 - f_3 - 2f_4 + t) - \epsilon_2 + \epsilon_3$$

$$f_4' = f_4(2f_1 + 2f_3 + f_4 - t) - \epsilon_3 + \epsilon_4$$

The general case for odd n is similar. Even n is slightly different.

33. Besides the two hierarchies there is an interesting Zoo of Connections for Isomonodromy. This is work in progress.

THANK YOU FOR YOUR ATTENTION